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Fractal Dimensions for Repellers of Maps with Holes

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In this work we study the Hausdorff dimension and limit capacity for repellers of certain non-uniformly expanding maps f defined on a subset of a manifold. This subset is covered by a finite number of compact domains with pairwise disjoint interiors (the complement of the union of these domains is called *hole*) each of which is mapped smoothly to the union of some of the domains with a subset of the hole. The maps are *not* assumed to be hyperbolic nor conformal. We provide conditions to ensure that the limit capacity of the repeller is less than the dimension of the ambient manifold. We also prove continuity of these fractal invariants when the volume of the hole tends to zero.

KEY WORDS: Hausdorff dimension; limit capacity; fractal dimensions; maps with holes; non-uniformly expanding maps; volume comparison method.

1. INTRODUCTION

1.1. Background and Motivation

Fractal invariants such as the Hausdorff dimension have been an important topic in many branches of Dynamical Systems. They have been used in topological, geometric and ergodic approaches to Dynamics, providing information about the dynamical behavior of maps and describing the geometrical structure of invariant sets. In the thermodynamical formalism they constitute a beautiful bridge between geometrical aspects and physical concepts such as entropy. There is now a rich theory of fractal dimensions for invariant sets of uniformly hyperbolic systems, especially in the case of surfaces. For instance,[(9)] provides a formula for the dimension of horseshoes from which one gets that the Hausdorff dimension and the limit capacity depend continuously on the dynamics, and are strictly less than

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2, at least when the dynamics is C^2. For more information see ref. 10, chapter 4. In this work we extend some of the conclusions of this theory beyond the uniformly hyperbolic setup.

In a d-dimensional manifold, consider M a set which is the closure of its interior. Suppose that f is a map defined on finitely many, compact domains $R_i \subset M$ with pairwise disjoint interiors, whose union is not the whole M. The subset of M where f is not defined is called hole. Each domain R_i is mapped onto a subset of M which is the union of some of the domains with, possibly, some part of the hole. Moreover we suppose that f is expanding in a non-uniform fashion to be made precise later. The main object of interest is the set Λ of points that never fall into the hole (the repeller) under iteration. Our first main result states that *limit capacity of* Λ *is less than the dimension of the ambient manifold*. In ref. 8, Horita and Viana proved for a setting very similar to ours, the analogous result on Hausdorff dimension. It is known that Hausdorff dimension is less or equal than limit capacity, so their result is included in ours. Since our methods are very different, the present paper gives, in particular, a new proof to the main result in ref. 8. A second set of new results presented in this work is the dependence of Hausdorff dimension and limit capacity on the Lebesgue measure of the hole. We prove that *the Hausdorff dimension and the limit capacity of the repeller converge to the dimension of the ambient manifold if the Lebesgue measure of the hole goes to zero*. Clearly the result on limit capacity follows from the one on Hausdorff dimension. Nevertheless we provide a different proof to the former, whose main advantage resides on its simplicity. One interesting setting where these results can be applied is the class of diffeomorphisms derived from Anosov diffeomorphisms through a Hopf bifurcation. See refs. 1, 7. Using[7] it follows from our results that *the limit capacity and the Hausdorff dimension of repellers derived from Anosov diffeomorphisms through Hopf bifurcations is less than the dimension of the ambient manifold*. Furthermore, *both fractal dimensions converge to the dimension of the manifold when the map converges to the bifurcating diffeomorphism*. Our conclusions may be seen as an extension of results of Diaz and Viana in ref. 6. where they considered diffeomorphisms derived from Anosov on the two-dimensional torus. A fundamental difference is that our repellers are *neither* uniformly hyperbolic *nor* conformal along invariant directions. Let us comment a bit more on this. Most results on fractal dimensions of invariant sets of dynamical system rely on auto-similarity, which on its turn, is often derived from hyperbolicity and conformality. The first of these properties is used to ensure control of volumes proportions (through distortion arguments) while conformality provides control of shapes of removed sets (pre-images of the hole). Neither of these two properties holds in our setting. Our strategy is to approach fractal dimensions through volume estimates. To begin with, we are able to obtain volume distortion control with our assumption about non-uniform hyperbolicity. Moreover, this approach also allows us to bypass the lack of conformality: for the limit capacity statement, besides the volume estimates we only need some control of limit capacity of pre-images of the boundary of the hole, which is obtained following an argument from ref. 8. On the other hand, it is a bit more delicate to avoid using conformality when dealing with Hausdorff dimension. To go around this problem we propose an approach we call Volume Comparison Method. Our key ingredient is a control of the spatial distribution of the removed sets, which we obtain using the structure of pre-images of the domains where the map is defined by the inverse branches, more precisely, controlling the diameters of such pre-images.

Let us also mention that ergodic properties of Anosov maps with holes have been studied by Chernov, Markarian and Troubetzkoy (see refs. 2–5). For an account on fractal dimensions and dynamical systems we shall refer to the recent Pesin monograph.[11]

1.2. Definitions and Results

1.2.1. Map with Hole

Consider $\mathcal{M}$ a compact d-dimensional Riemannian manifold. Let $M \subset \mathcal{M}$ be a set that is the closure of its interior and $R_1, \dots, R_m$ be domains in M whose interiors are pairwise disjoint (we use the term *domain* to mean a compact path-connected set) and such that the limit capacity of the boundary of R_i is less than d for all i. The inner diameter of a subset of $\mathcal{M}$ is the supremum of the inner distances between any two points in the same connected component, where the inner distance is the infimum of all lengths of curves connecting the two points. A *map with hole* is a map $f : R_1 \cup \dots \cup R_m \to M$ such that $\log|\det(Df^{(-1)})|$ is (C_0, ϵ)-Hölder continuous for some $C_0 > 0$ and $\epsilon > 0$, for any inverse branch $f^{(-1)}$ of f and the restriction of f to each R_i is a diffeomorphism onto some domain W_i with the following properties:

- for all $i \in \{1, \dots, m\}$, there exists some $j \in \{1, \dots, m\}$ such that $R_j \subset W_i$;
- $R_j \subset W_i$ whenever $R_j \cap W_i \neq \emptyset$;
- for all $i \in \{1, \dots, m\}$, W_i have finite inner diameter (let us call ρ the maximum inner diameter of W_i over $i \in \{1, \dots, m\}$);
- Markovian property: the union of the boundaries of R_j for $j \in \{1, \dots, m\}$ is an invariant set, that is, if x belongs to the boundary of any R_i, its image also belongs to the boundary of some R_j.

Notation 1. Defining $D = \min\{\text{Leb}(W_i) : i = 1, \dots, m\}$ we say that f is a (D, ρ, C_0, ϵ)-map with hole according to the constants specified above.

Remark 1. The condition on $\log|\det(Df^{(-1)})|$ is satisfied if, for instance, f is $C^{1+\epsilon}$.

The set $H_f = M \backslash (R_1 \cup \dots \cup R_m)$ is the *hole of* f and we call ρ the inner diameter of f. The repeller of f is the set Λ_f of points in M whose forward orbits never enter H_f:

$$\Lambda_f = \{x \in M : f^n(x) \in R_1 \cup \dots \cup R_m \text{ for every } n \geqslant 0\}$$

If we define

$$\Lambda_n = \{x \in M : f^j(x) \in R_1 \cup \dots \cup R_m \text{ whenever } 0 \leqslant j < n\}$$

it follows that $\Lambda_f = \cap_{n \in \mathbb{N}} \Lambda_n$.

1.2.2. Expanding and Non-Uniformly Expanding Maps

Given $c > 0$, a local diffeomorphism F is called *c-expanding* if there exist n such that the derivative of any inverse branch satisfies $||DF^{(-n)}(x)|| < e^{-nc}$ for all x in its domain of definition. For our purposes, that is, the study of the repeller, it makes no difference whether we deal with the original map or with a fixed iterate (that is, to work with F^n instead of F). Therefore, to simplify our presentation, we shall assume $n = 1$. In our paper, the notation F is used here to maps known to be expanding, while f is usually applied to what we call a non-uniformly expanding map, according to the definition bellow. Given a local diffeomorphism f we denote by $f^{(-1)}$ any inverse branch of f. Given $n \geqslant 1$ and $\alpha_1, \dots, \alpha_n$ in $\{1, \dots, m\}$, a set

$$C(\alpha_1, \dots, \alpha_n) = R_{\alpha_1} \cap f^{-1}(R_{\alpha_2}) \cap \dots \cap f^{-(n-1)}(R_{\alpha_n})$$

is called and *n-cylinder* for f.

For a map with hole f, we define the function

$$\phi_n(\alpha_1, \dots, \alpha_n) = \frac{1}{n} \sum_{j=1}^{n} \inf_{x \in C_j} \log ||Df^{-1}(f^j(x))||^{-1}$$

where $C_j = C(\alpha_1, \dots, \alpha_j)$. We say that ϕ_n is the *average least expansion*. Obviously, if f is immediately c-expanding, that is, if $||Df^{-1}(x)|| \leqslant e^{-c}$

for all x, then $\phi_j(\alpha_1,\dots,\alpha_j)\geqslant c$ for all j. Let us call $\mathcal{S}_0(c)$ the set of points that belong to some cylinder $C(\alpha_1)$ such that $\phi_1(\alpha_1)>c$ and $\mathcal{B}_n(c)$ the set of points that belong to some cylinder $C(\alpha_1,\dots,\alpha_n)$ such that $\phi_j(\alpha_1,\dots,\alpha_j)\leqslant c$ for all $j\in\{1,\dots,n\}$. We notice that the set $\mathcal{B}_n(c)$ is a union of cylinders. From now on we will say that n is a c-expanding time for a cylinder $C_j=C(\alpha_1,\dots,\alpha_j)$ meaning that $\phi_j(\alpha_1,\dots,\alpha_j)\geqslant c$.

The definition of a non-uniformly c-expanding map requires that the measure of cylinders that take a long time to have an expanding time decays in a particular fashion. Let $\{\delta_n(c)\}_{n\in\mathbb{N}}$ be a sequence of real positive numbers converging to zero. We say that a map f has *c-decay* $\delta_n(c)$ if $\mathrm{Leb}(\mathcal{B}_n(c))\leqslant\delta_n(c)$ for all $n\geqslant 1$. Since c will be fixed through most of this work, we refer to c-decay just as *decay*. Given $c\geqslant 0$, we say that f is non-uniformly c-expanding if it satisfies

(NU_1) There exists a sequence $\delta_n(c)$ going to zero such that $\mathrm{Leb}(\mathcal{B}_n(c))\leqslant\delta_n(c)$.

We say that a non-uniformly c-expanding map f has exponential decay if

(NU_2) There exists $c_1>0$ such that, for every large n, we have $\mathrm{Leb}(\mathcal{B}_n(c))\leqslant e^{-c_1 n}$.

Any c-expanding map satisfies assumptions (NU_1) and (NU_2).

We say that a family $\mathcal{F}$ of non-uniformly c-expanding maps has common decay if

(NU_3) There exists a sequence $\delta_n(c)$ going to zero such that for every $f\in\mathcal{F}$ we have $\mathrm{Leb}(\mathcal{B}_n(c))\leqslant\delta_n(c)$.

1.2.3. Fractal Dimensions

For $\alpha\geqslant 0$, the Hausdorff α-measure of a metric space X is defined by

$$m_\alpha(X)=\lim_{\epsilon\to 0}\inf\left\{\sum_{U\in\mathcal{U}}(\mathrm{diam}U)^\alpha : \mathcal{U}\text{ is an open covering of } X \text{ with } \mathrm{diam}U\leqslant\epsilon \text{ for all } U\in\mathcal{U}\right\}.$$

It is easy to show that there exists a unique number $HD(X)$, called *Hausdorff dimension of* X, such that $m_\alpha(X)=\infty$ for any $\alpha<HD(X)$ and $m_\alpha(X)=0$ for any $\alpha>HD(X)$. The *limit capacity* or *box dimension*, of a metric space X is defined by

$$Cap(X) = \limsup_{\epsilon \to 0} \frac{\log n(X,\epsilon)}{|\log \epsilon|},$$

where $n(X,\epsilon)$ is the smallest number of ϵ-balls needed to cover X.

1.2.4. Results

Our first main theorem provides an upper bound for the limit capacity of Λ_f, which is strictly smaller than the Hausdorff dimension of M, when f is a map with hole satisfying (NU_2).

Theorem 1. Consider f a non-uniformly c-expanding map with hole with exponential decay. Moreover suppose that the difference $H_i = W_i \backslash (R_1 \cup \cdots \cup R_m)$ has non-empty interior for all i. Then $Cap(\Lambda_f) < d$ if and only if $Cap(\partial \cup_i H_i) < d$.

Remark 2. For the theorem above the Markovian property is necessary for the only if part but not for the direct implication, that is, as we can notice along the proofs, if f doesn't have the Markovian property but except by this lack it fits on the hypothesis of the theorem, we still verify that $Cap(\partial \cup_i H_i) < d$ implies $Cap(\Lambda_f) < d$. It is easy to see how the Markovian property is used in the only if part: it implies that $\partial H_i \subset \cup \partial R_i \subset \Lambda_f$. Therefore, with this property, if $Cap(\partial H_i) = d$ then $Cap(\Lambda_f) \geqslant d$.

A second set of new results provides lower bounds for the limit capacity and Hausdorff dimension of Λ_f depending on the volume of the hole. We use them to prove the continuity of such invariants when the Lebesgue measure of the hole tends to zero. That is, if Lebesgue measure of the hole tends to zero, the fractal dimensions tend to the dimension of the manifold.

More precisely, we prove that, having fixed a few technical constants (namely, the expansion constant c, the type $\delta_n(c)$ of decay the constants of Hölder continuity (C_0, ϵ), an upper bound to $||Df^{(-1)}||$, the number m of inverse branches and an upper bound for the inner diameter and a lower bound to Lebesgue measure of the domains W_i) then there are lower bounds for the Hausdorff dimension and limit capacity of its repeller depending only on Lebesgue measure of H_f. Moreover, these bounds imply that when $\mathrm{Leb}(H_f)$ tends to zero, the Hausdorff dimension and the limit capacity estimates converge to d. Although the last convergence follows directly from the former (recall the well-known inequality $HD(X) \leqslant$

$Cap(X)$) we provide an alternative and much simpler proof to the result on limit capacity.

These statements are contained in Theorems 2, 3, and 4. Theorems 2 and 3 provide respectively a lower bound to limit capacity and Hausdorff dimension of the repeller as functions of Lebesgue measure of the hole in the case of expanding maps with hole. Theorem 4 allows us to apply those theorems also to the case of non-uniformly expanding maps by the construction of an immediately expanding map with repeller contained in the non-uniform one, and such that the measure of the hole is a function of the measure of the hole of the original map, converging to zero when the latter converges to zero.

Theorem 2. Suppose that $\mathcal{M}$ is d-dimensional. Given $c>0$, $\epsilon>0$, $\rho>0$, $D>0$ and a constant C_0', there exists a map

$$\psi_0:[0,1]\to[0,d]$$

such that $\psi_0(x)$ converges to d when x converges to zero and for any c-expanding (D,ρ,C_0',ϵ)-map with hole F we have that

$$Cap(\Lambda_F)\geqslant\psi_0(\mathrm{Leb}(H_F)).$$

Theorem 3. Suppose that $\mathcal{M}$ is d-dimensional. Given $c>0$, $\epsilon>0$, $\rho>0$, $D>0$ and a constant C_0', there exists a map

$$\psi_1:[0,1]\to[0,d]$$

such that $\psi_1(x)$ converges to d when x converges to zero and for any c-expanding (D,ρ,C_0',ϵ)-map with hole F we have that

$$HD(\Lambda_F)\geqslant\psi_1(\mathrm{Leb}(H_F))$$

Theorem 4. Given (C_0,ϵ), $c>0$, $S>0$, $\rho>0$, $D>0$, $m\in\mathbb{N}$, and a sequence $\delta_n(c)$, there exists a constant C_0' and a function $\psi_2:[0,1]\to[0,1]$ satisfying

$$\lim_{x\to 0}\psi_2(x)=0$$

such that if f is a non-uniformly c-expanding (D,ρ,C_0,ϵ)-map with hole with the following properties:

(a) f has not more than m inverse branches (not more than m domains R_j);

(b) $\sup ||Df^{(-1)}|| \leqslant S$;

then there exists a c-expanding $(D, \rho, C_0', \epsilon)$-map with hole F in $\mathcal{M}$ whose repeller is contained in the repeller of f and such that $\text{Leb}(H_F) < \psi_2(\text{Leb}(H_f))$.

Fixing the constants (C_0, ϵ), $c > 0$, $S > 0$, $\rho > 0$, $m \in \mathbb{N}$ and D, consider a family of non-uniformly c-expanding (D, ρ, C_0, ϵ)-maps with hole $\{f_\alpha\}_\alpha$, such that all map in the family satisfies (a) and (b) with the fixed constants. Moreover suppose that the family have common decay. For each α, let H_α be the hole of f_α and Λ_α be its repeller. In this setting Theorems 2, 3 and 4 imply the following result:

Corollary 5. $HD(\Lambda_\alpha)$ and $Cap(\Lambda_\alpha)$ converge to d when $\text{Leb}(H_\alpha)$ tends to zero.

Considering that M is a compact set in a Riemannian manifold it is covered by a finite number of charts. For simplicity we assume that M is contained in an d-dimensional unitary cube K (a *d-cube*), what is equivalent to consider M covered by only one chart. The general case is analogous.

2. AUXILIARY NOTIONS AND RESULTS

2.1. Squared Partitions

Along all the next sections we use the following simple notation:

Definition 1. Given natural numbers k and n, the *k-square partition of order n* is the partition of the d-cube in k^{dn} disjoint cubes with volume $1/k^{dn}$. Each one of this cubes is called an *n-element*. We say that n is the *level of the element*.

2.2. Induced Map and Bounded Volume Distortion Tools

The first step is to control the volume withdrawn at each step of the construction of the repeller. This is done through a bounded distortion argument using an expanding map induced from f. Consider $\mathcal{S}_n(c) = \mathcal{B}_n(c) \backslash (\mathcal{B}_{n+1}(c) \cup f^{(-n)}(H_f))$ for all $n \geqslant 1$ (we recall that $\mathcal{S}_0(c)$ was defined at introduction). We notice that the sets $\mathcal{S}_n(c)$ are disjoint and for $n \geqslant 0$ the set $\mathcal{S}_n(c)$ is a union of $(n+1)$-cylinders, thus f^{n+1} is defined for $\mathcal{S}_n(c)$. Consider the map $F : \cup_{n \geqslant 0} \mathcal{S}_n(c) \to M$ defined by

$$F(x) = f^{n+1}(x) \quad \text{if } x \in \mathcal{S}_n(c).$$

It is easy to check that this is a c-expanding map. We state some distortion results for it. Except for minor points, next results are found in ref. 8.

Lemma 6. Given $C_0 > 0$, $c > 0$ and $\epsilon > 0$ there exists $C_0' > 0$ such that if $\log|\det(Df^{(-1)})|$ is (C_0, ϵ)-Hölder for any inverse branch $f^{(-1)}$ of f then $\log|\det(DF^{(-1)})|$ is (C_0', ϵ)-Hölder for any inverse branch $F^{(-1)}$ of F.

Proof. See the proof of Lemma 2.4 in ref. 8. ∎

Proposition 7 (*bounded distortion*). Let $C_1 = \exp(C_0' \sum_{j=0}^{\infty} e^{-cj\epsilon/2})$. Then

$$\frac{1}{C_1} \leqslant \frac{|\det DF^{(-n)}(y)|}{|\det DF^{(-n)}(z)|} \leqslant C_1$$

for every inverse branch $F^{(-n)}$ of F^n, any $n \geqslant 1$, and for every pair of points y, z in the domain of $F^{(-n)}$.

Proof. See the proof of Proposition 2.5 in ref. 8. ∎

Corollary 8. Let $C_2 = C_1^2$. Then, given $n \geqslant 1$ and any inverse branch $F^{(-n)}$ of F^n, we have

$$\frac{1}{C_2} \frac{\mathrm{Leb}(A)}{\mathrm{Leb}(B)} \leqslant \frac{\mathrm{Leb}(F^{(-n)}(A))}{\mathrm{Leb}(F^{(-n)}(B))} \leqslant C_2 \frac{\mathrm{Leb}(A)}{\mathrm{Leb}(B)}$$

for any measurable subsets A and B of the domain of $F^{(-n)}$.

Proof. See the proof of Corollary 2.6 in ref. 8. ∎

In our settings it will be useful the following particularization of the corollary above:

Corollary 9. Given $n \geqslant 1$, any inverse branch $F^{(-n)}$ of F^n and measurable subsets A and B of the domain of $F^{(-n)}$ such that $B \subset A$ we have

$$\mathrm{Leb}(F^{(-n)}(A) \setminus F^{(-n)}(B)) \geqslant \mathrm{Leb}(F^{(-n)}(A)) \left(1 - C_2 \frac{\mathrm{Leb}(B)}{\mathrm{Leb}(A)}\right).$$

Proof. Since $B \subset A$ we have that

$$\mathrm{Leb}(F^{(-n)}(A) \setminus F^{(-n)}(B)) = \mathrm{Leb}(F^{(-n)}(A)) - \mathrm{Leb}(F^{(-n)}(B)).$$

Moreover Corollary 8 implies that

$$\mathrm{Leb}(F^{(-n)}(B)) \leqslant C_2 \frac{\mathrm{Leb}(B)}{\mathrm{Leb}(A)} \mathrm{Leb}(F^{(-n)}(A))$$

completing the proof. ∎

An important fact to be noticed here is that the constants C_1 and C_2 depend only on the expansivity constant c and on the Hölder continuity constants C_0 and ϵ, they do not depend on f (or F) itself.

3. UPPER BOUND FOR THE LIMIT CAPACITY

Our goal in this section is to prove Theorem 1.

3.1. Idea of the Proof

To explain the idea behind the argument, we first observe some facts about a very simple fractal set. Let Q_0 be a square with unitary side. Split Q_0 in four equal squares and remove the subset on the top and left. Let Q_1 be the set composed by the three remaining squares. Repeat this process with each one of the three squares to obtain Q_2. Let Q be the fractal set obtained as the intersection of the sets remaining at each step if this operation is repeated infinitely many times.

Clearly, for any n natural, we are able to cover Q_n using 3^n squares with sides measuring $(1/2)^n$, rewriting, we would do it with $(1/2)^{-n\log_2 3}$ such squares.

Let us see that this intuitive property is sufficient to prove that limit capacity of Q is less or equal to $\log_2 3$. Stating more precisely,

Lemma 10. Let Q be a set. If there exists $\epsilon < 1$ such that for any $n > 0$ there exists a covering of Q composed by ϵ^{-nd} balls with ratio ϵ^n then the limit capacity of Q is less or equal to d.

Proof. According to the definition of limit capacity, we have to prove that

$$\mathrm{Cap}(Q) = \limsup_{\delta \to 0} \frac{\log n(Q, \delta)}{|\log \delta|} \leqslant d,$$

where $n(Q, \delta)$ is the smallest number of δ-balls needed to cover Q.

For δ small enough (smaller than the fixed ϵ), there exists m such that $\delta = \epsilon^{m+\theta}$ where $\theta \in (0, 1]$. Therefore, if there is a covering of Q by $\epsilon^{-(m+1)d}$

balls of ratio ϵ^{m+1}, certainly there exists a covering of Q by $\epsilon^{-(m+1)d}$ balls of ratio δ (since $\delta \geqslant \epsilon^{m+1}$). It follows that

$$\frac{\log n(Q,\delta)}{|\log \delta|} \leqslant \frac{\log \epsilon^{-(m+1)d}}{|\log \epsilon^{m+\theta}|} = \frac{(m+1)d}{m+\theta}.$$

When δ converges to zero, m grows without bounds and last fraction converges to d, proving the result stated. ∎

Essentially, in the example of the classical fractal set mentioned above, what allows us to easily state the existence of the desired covering is the fact that the sets Q_n, themselves, can be covered in such fashion. The point is that they are very simple non-fractal sets and we are able to deal easily with them. But once we have covered Q_n, we have the covering of Q since $Q \subset Q_n$.

The argument that we develop now is very similar to this one. We prove that there are coverings to Λ_n with the properties stated in the lemma above, and using this lemma we conclude that Λ_f has limit capacity less or equal to a $d' < d$.

In order to verify if it is possible to cover Λ_n as we want, the first attempt is to consider the volume problem dividing Leb(Λ_n) by $\epsilon^{-d'n}$. Here we face the first difficulty related to the lack of hyperbolicity: it is hard to estimate Leb(Λ_n) since we don't have control of distortion. If we had uniform expansion we could use a bounded distortion argument and find $\eta < 1$ such that Leb$(\Lambda_n) < \eta^n$, what would be very nice since it would give an affirmative answer to our first test: the volume of Λ_n would be lower than the volume of $\epsilon^{-d'n}$ balls with radius ϵ^n (if d' satisfies $d - d' \leqslant \log \eta / \log \epsilon$). Let us suppose for a while that we have Leb$(\Lambda_n) < \eta^n$. This estimate does not assure the existence of an ϵ^n-covering of Λ_n with no more than $\epsilon^{-d'n}$ balls, but it ensures that we have such a covering of $\Lambda_n \backslash \Lambda_n^{\epsilon\partial}$, where $\Lambda_n^{\epsilon\partial}$ is an ϵ^n-neighborhood of the boundary of Λ_n.

Therefore the work to find the upper bound to the limit capacity will be split in two parts: first, to prove that even in our non-uniformly hyperbolic situation we have the estimate Leb$(\Lambda_n) < \eta^n$, and then to show that we can cover $\Lambda_n^{\epsilon\partial}$ with a small number of ϵ^n-balls. The second part consists in an adaptation of arguments of Proposition 4.1 in ref. 8. The former is done through the observation that the volume removed at each step from the cylinders where we assure the control of distortion is large enough to compensate the uncertainty about the volume removed where we don't have such a control.

3.2. Proof of Theorem 1

Lemma 11 provides a way to establish un upper bound for the limit capacity of a fractal set obtained as limit of a sequence of nested sets requiring conditions only on the nested sets (not on the fractal set itself). Lemmas 12 and 13 assure that the nested sets that generate the repeller of a map with hole considered on Theorem 1 satisfy these conditions.

Definition 2. Given a sequence of nested sets $(\Gamma_n)_{n\in\mathbb{N}}$, we say that it has d-non-significant boundary if given ϵ small enough exists $d'<d$ such that for all $j\in\mathbb{N}$ we can find a covering of the ϵ^j-neighborhood of Γ_j by $\epsilon^{-d'j}$ balls of radius ϵ^{-j}.

Lemma 11. If $(\Gamma_n)_{n\in\mathbb{N}}$ is a sequence of nested sets with d-non-significant boundary such that $\mathrm{Leb}(\Gamma_j)\leqslant\eta^j$ for some $\eta\in(0,1)$ then the limit set $\Gamma=\cap_n\Gamma_n$ has limit capacity less than d.

Proof of Lemma 11. For all natural number k the amount of elements of the k-square partition of order n that are contained inside Γ_n is less or equal than $\eta^n k^{nd}$, that is,

$$k^{n\left(d+\frac{\log\eta}{\log k}\right)}.$$

These elements cover $\Gamma_n\backslash\Gamma_n^\partial$, where Γ_n^∂ is the intersection of a $1/k^n$-neighborhood of the boundary of Γ_n with Γ_n. The d-non-significant boundary hypothesis ensures that there is some $d'<d$ such that if $1/k$ is small enough we can cover Γ_n^∂ with $k^{d'n}$ balls of radius $1/k^n$. It follows that for each n there is a $1/k^n$-covering of Γ with not more than $k^{n\left(d+\frac{\log\eta}{\log k}\right)}+k^{d'n}$, so $\mathrm{Cap}(\Gamma)\leqslant\max\{d',d+\frac{\log\eta}{\log k}\}<d$. ∎

Now we turn to our setting of maps with hole and state

Lemma 12. If $Cap(\partial H_f)<d$, then the nested sequence Λ_n has d-non-significant boundary.

Lemma 13. There exist $\eta\in(0,1)$ such that $\mathrm{Leb}(\Lambda_j)\leqslant\eta^j$ for all $j\in\mathbb{N}$.

Proof of Theorem 1. Since, by Markovian property, $\partial H_i\subset\Lambda_f$, it is obvious that $\mathrm{Cap}(\partial H_i)\geqslant d$ implies $\mathrm{Cap}(\Lambda_f)\geqslant d$. Lemma 11 ensures that if the sequence Λ_n has d-non-significant boundary and there is $\eta\in(0,1)$ such that $\mathrm{Leb}(\Lambda_j)\leqslant\eta^j$ for all $j\in\mathbb{N}$, then it follows that $\mathrm{Cap}(\Lambda_f)<d$. These conditions are verified in Lemma 12 and Lemma 13. ∎

3.3. Proof of Lemma 12

Proposition 14. Let R be a domain in a d-dimensional manifold M such that the limit capacity of the boundary of R is $d_0 < d$. Consider any $d_1 \in (d_0, d)$ and $g : M \to M$ a local diffeomorphism. Given ϵ small enough there exists C such that for any $n \in \mathbb{N}$ there is a covering of the ϵ^n-neighborhood of $g^n(\partial R)$ by ϵ^n-balls with no more than $CK^{n(2d-d_1)}\epsilon^{-nd_1}$ elements ($K > 1$ is an upper bound to $||Dg||$ and $||Dg^{-1}||$).

This proposition is an adaptation of Proposition 4.1 from Horita–Viana[8] and is proved along the same lines:

Proof. Given $\epsilon > 0$ small enough there is a covering of ∂R by balls with radius $K^n\epsilon^n$, with no more than $C_1(K\epsilon)^{-d_1 n}$ balls $B(x_i, K^n\epsilon^n)$ (let us fix the points x_i). The images of these balls by g^n are contained inside balls $B(g^n(x_i), K^{2n}\epsilon^n)$. Let us consider a covering of these images by the balls $B(g^n(x_i), 2K^{2n}\epsilon^n)$. We will verify that these balls cover an ϵ^n-neighborhood V of $g^n(\partial R)$. Given $y \in V$, there exists $x \in \partial R$ such that $d(y, x) < \epsilon^n$, and then $d(g^{-n}(x), g^{-n}(y)) < K^n\epsilon^n$. It follows that there is $x_i \in \partial R$ such that $d(g^{-n}(y), x_i) < 2K^n\epsilon^n$, and this implies $d(y, g^n(x_i)) < 2K^{2n}\epsilon^n$. Since M is a manifold with bounded curvature there exists C_2 such that each ball $B(g^n(x_i), 2K^{2n}\epsilon^n)$ may be covered by C_2K^{2nd} balls with radius ϵ^n, that is, we have the covering we were looking for with no more than $CK^{n(2d-d_1)}\epsilon^{-nd_1}$ elements. ∎

Proof of Lemma 12. Fix $d_2 \in (d_1, d)$. The boundary of Λ_n is composed by the union of boundaries of H_i and their pre-images (by $n-1$ iterations). By hypothesis, for each H_i we have no more than $\sum_{i=0}^{n-1} m^i$ pre-images (H itself is here considered as pre-image of order 0). This number is upper bounded by p^n for some p fixed. Therefore, according to the last proposition, we have a covering of the ϵ^n-neighborhood of the boundary of Λ_n by no more than $Cp^nK^{n(2d-d_1)}\epsilon^{-nd_1}$ balls.

$$Cp^nK^{n(2d-d_1)}\epsilon^{-nd_1} = C(pK^{2d-d_1}\epsilon^{-d_1})^n \leqslant C\epsilon^{-d_2 n}$$

if

$$-\log\epsilon \geqslant \frac{1}{d_2 - d_1}(2(d-d_1)\log K + \log p)$$

This proves the lemma. ∎

3.4. Proof of Lemma 13: Reorganizing Trajectories

Fix c in hypothesis (NU_1). The first idea we develop for controlling the volume removed at each step of the construction of Λ_f is the observation that what we really have to control is the trajectory of points considering a partition of M in sets $\mathcal{S}_n(c)$ instead of the partition into domains $\{R_1, \ldots, R_m\}$. In this direction we shall now group cylinders that have the same behavior with respect to $\mathcal{S}_n(c)$. Define

$$G_n(\beta^n) = \mathcal{B}_n(c)$$

and

$$G_n(\beta^{n-1}\alpha) = \mathcal{S}_n(c)$$

for $n \geqslant 1$ (observe that last definition implies $G_1(\alpha) = \mathcal{S}_0(c)$). We use β^n to denote that the point fell inside a set where the first expanding time is bigger than n, while α assures the existence of an expanding time. We use the identity $\beta^n\beta = \beta^{n+1}$ in our notation.

Once defined $G_n(\beta^n)$ and $G_n(\beta^{n-1}\alpha)$, given any

$$\gamma^n \in \Gamma_n = \{\gamma_1\gamma_2 \ldots \gamma_n \text{ where } \gamma_i \in \{\alpha, \beta\} \text{ for all } i\},$$

we define $G_n(\gamma^n)$ by induction. Suppose we have $G_n(\gamma^n)$ for $\gamma^n = \gamma_1\gamma_2 \ldots \gamma_{n-1}\alpha$. It is enough to define $G_{n+k}(\gamma^n\beta^k)$, $G_{n+1}(\gamma^n\alpha)$ and $G_{n+k+1}(\gamma^n\beta^k\alpha)$:

$$G_{n+k}(\gamma^n\beta^k) = \{x \in G_n(\gamma^n) \text{ such that } f^n(x) \in G_k(\beta^k)\}$$

$$G_{n+k+1}(\gamma^n\beta^k\alpha) = \{x \in G_n(\gamma^n) \text{ such that } f^n(x) \in G_{k+1}(\beta^k\alpha)\}$$

$$G_{n+1}(\gamma^n\alpha) = \{x \in G_n(\gamma^n) \text{ such that } f^n(x) \in G_1(\alpha)\}.$$

Notice that each $G_n(\gamma^n)$ is a union of n-cylinders and we have

$$\Lambda_n = \cup_{\gamma^n \in \Gamma_n} G_n(\gamma^n)$$

Recall the map F defined in Section 2.2. It is easy to see that

$$F(x) = f^n(x) \quad \text{for all } x \in G_n(\beta^{n-1}\alpha) \text{ for } n \geqslant 1.$$

Lemma 15. Consider $\gamma^n = \alpha^{a_1}\beta^{b_1}\alpha^{a_2}\ldots\beta^{b_k}\alpha^{a_{k+1}}$, where $\sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k} b_i = n$ and a_i and b_i are integers bigger than zero except, possibly, a_1, that might be zero. Then we have

$$f^n|_{G_n(\gamma^n)} = F^N|_{G_n(\gamma^n)}, \quad \text{where } N = k + \sum_{i=1}^{k+1} a_i.$$

Proof. It is easy to verify that

$$f(G_n(\alpha\gamma^{n-1})) \subset G_{n-1}(\gamma^{n-1}) \text{ and } f^{k+1}(G_{n+k+1}(\beta^k\alpha\gamma^n)) \subset G_n(\gamma^n). \quad (1)$$

Moreover $G_n(\alpha^a\gamma^{n-a}) \subset G_a(\alpha^a)$ where $f^a = F^a$, so

$$f^a|_{G_n(\alpha^a\gamma^{n-a})} = F^a|_{G_n(\alpha^a\gamma^{n-a})}.$$

Considering (1) now it is enough to see that

$$f^{b+1}|_{G_n(\beta^b\alpha\gamma^{n-b-1})} = F|_{G_n(\beta^b\alpha\gamma^{n-b-1})},$$

what follows from the definition of F since $G_n(\beta^b\alpha\gamma^{n-b-1}) \subset G_{b+1}(\beta^b\alpha)$. ∎

Let us call $S_n(\gamma^{n-1}\beta)$ the set $G_n(\gamma^{n-1}\beta)\setminus G_{n+1}(\gamma^{n-1}\beta^2)$. We see that $S_n(\gamma^{n-1}\beta)$ is the union of $G_{n+1}(\gamma^{n-1}\beta\alpha)$ with a pre-image of the hole by $f^{-(n+1)}$.

Last lemma states that the map f^n restricted to $G_n(\gamma^{n-1}\alpha)$ coincides with some iteration of the map F, what allows us to use the bounded distortion argument for f^n restricted to $G_n(\gamma^{n-1}\alpha)$. Therefore it is convenient to use the following partition of Λ_n:

$$\Lambda_n = \cup_{\gamma^{n-1}\in\Gamma_{n-1}}[G_n(\gamma^{n-1}\alpha) \cup (\cup_{j=1}^{\infty} S_{n+j-1}(\gamma^{n-1}\beta^j))] \quad (2)$$
$$\cup(\cap_{j=1}^{\infty} G_{n+j}(\gamma^{n-1}\beta^j)).$$

To find $\mathrm{Leb}(\Lambda_n)$ we just have to add together the measure of the sets on the right hand side of last equation. Due to the lack of hyperbolicity while iterating inside sets S_i it is not straightforward to show that $\mathrm{Leb}(\Lambda_n) < \eta^n$. Nevertheless, it turns out to be easy to prove a stronger fact: if, while computing the sum of the measures of those sets, we replace the terms $\mathrm{Leb}(S_i)$ by a special enlargement of them, the sequence L_n, obtained instead of (and greater than) $\mathrm{Leb}(\Lambda_n)$, has exponential decay. We now proposes a general formula for such a sequence L_n, depending on a constant ϵ related to the "enlargement" of S_i.

Consider the sequence

$$L_n = \sum_{\gamma^{n-1}\in\Gamma_{n-1}} \left[\mathrm{Leb}(G_n(\gamma^{n-1}\alpha)) + \left(\sum_{j=1}^{\infty} \frac{\mathrm{Leb}(S_{n+j-1}(\gamma^{n-1}\beta^j))}{(1-\epsilon)^j} \right) \right]$$

where a small enough $\epsilon \in (0,1)$ is fixed (the assumption of exponential decay assures that the sum converges provided that ϵ is small enough).

From the Eq. (2) it is clear that

$$\mathrm{Leb}(\Lambda_n) \leqslant L_n$$

(recall that the non-uniformly expanding assumptions implies that $\mathrm{Leb}(\cap_{j=1}^{\infty} G_{n+j}(\gamma^{n-1}\beta^j)) = 0$).

Therefore if we show that L_n decay exponentially, we will have a bound $\mathrm{Leb}(\Lambda_n) \leqslant \eta^n$ for some $\eta \in (0,1)$ and all natural n large enough

Next lemma shows that if ϵ in L_n formula is small enough then $L_{n+1} \leqslant L_n(1-\epsilon)$ (implying that L_n has exponential decay).

Lemma 16. Let $H = H_i$ for i such that $\mathrm{Leb}(H_i) \leqslant \mathrm{Leb}(H_j)$ for any $j = 1, \ldots, m$. If $\epsilon \in (0,1)$ satisfies

$$\mathrm{Leb}(H) \geqslant C_2\epsilon + C_2^2 \sum_{j=1}^{\infty} \left[\frac{\mathrm{Leb}(S_j(\beta^j))}{(1-\epsilon)^{j+1}} - \mathrm{Leb}(S_j(\beta^j)) \right] \tag{3}$$

where C_2 is the distortion constant from Corollary (8) in Section 2.2, then

$$L_n(1-\epsilon) \geqslant L_{n+1}.$$

Lemma 17. There is $\epsilon \in (0,1)$ satisfying (3).

Proof of Lemma 17. Since $C_2\epsilon$ can be made arbitrarily small, it is sufficient to prove that the same happens to

$$\sum_{j=1}^{\infty} \left[\frac{\mathrm{Leb}(S_j(\beta^j))}{(1-\epsilon)^{j+1}} - \mathrm{Leb}(S_j(\beta^j)) \right]. \tag{4}$$

Let us fix $\delta > 0$ and show that if ϵ is small enough then (4) is lower than δ. By hypothesis $(NU_2)\mathrm{Leb}(S_j(\beta^j))$ decays exponentially. Therefore there is ϵ_0 such that if $0 < \epsilon < \epsilon_0$ then

$$\frac{\mathrm{Leb}(S_j(\beta^j))}{(1-\epsilon)^{j+1}}$$

also decays exponentially and then there is N such that

$$\sum_{j=N}^{\infty} \frac{\mathrm{Leb}(S_j(\beta^j))}{(1-\epsilon)^{j+1}} < \frac{\delta}{4}.$$

Clearly for this N we also have $\sum_{j=N}^{\infty} \mathrm{Leb}(S_j(\beta^j)) < \delta/4$. Now just take ϵ_1 such that if $0 < \epsilon < \epsilon_1$ then

$$\sum_{j=1}^{N-1} \left[\frac{\mathrm{Leb}(S_j(\beta^j))}{(1-\epsilon)^{j+1}} - \mathrm{Leb}(S_j(\beta^j)) \right] < \frac{\delta}{2}.$$

Any $\epsilon < \min\{\epsilon_0, \epsilon_1\}$ will turn (4) less then δ. ∎

Proof of Lemma 16.

$$L_n = \sum_{\gamma^{n-1} \in \Gamma_{n-1}} \left[\mathrm{Leb}(G_n(\gamma^{n-1}\alpha)) + \sum_{j=1}^{\infty} \frac{\mathrm{Leb}(S_{n+j-1}(\gamma^{n-1}\beta^j))}{(1-\epsilon)^j} \right]$$

$$L_{n+1} = \sum_{\gamma^n \in \Gamma_n} \left[\mathrm{Leb}(G_{n+1}(\gamma^n\alpha)) + \sum_{j=1}^{\infty} \frac{\mathrm{Leb}(S_{n+j}(\gamma^n\beta^j))}{(1-\epsilon)^j} \right]$$

$$= \sum_{\gamma^{n-1} \in \Gamma_{n-1}} \left\{ \mathrm{Leb}(G_{n+1}(\gamma^{n-1}\alpha^2)) + \mathrm{Leb}(G_{n+1}(\gamma^{n-1}\beta\alpha)) + \sum_{j=1}^{\infty} \left[\frac{\mathrm{Leb}(S_{n+j}(\gamma^{n-1}\alpha\beta^j))}{(1-\epsilon)^j} + \frac{\mathrm{Leb}(S_{n+j}(\gamma^{n-1}\beta^{j+1}))}{(1-\epsilon)^j} \right] \right\}.$$

It is enough to show that

$$\mathrm{Leb}(G_n(\gamma^{n-1}\alpha))(1-\epsilon) \geqslant \mathrm{Leb}(G_{n+1}(\gamma^{n-1}\alpha^2)) + \sum_{j=1}^{\infty} \frac{\mathrm{Leb}(S_{n+j}(\gamma^{n-1}\alpha\beta^j))}{(1-\epsilon)^j} \tag{5}$$

and

$$\sum_{j=1}^{\infty} \frac{\mathrm{Leb}(S_{n+j-1}(\gamma^{n-1}\beta^{j}))}{(1-\epsilon)^{j}}(1-\epsilon) \geqslant \mathrm{Leb}(G_{n+1}(\gamma^{n-1}\beta\alpha)) + \sum_{j=1}^{\infty} \frac{\mathrm{Leb}(S_{n+j}(\gamma^{n-1}\beta^{j+1}))}{(1-\epsilon)^{j}} \tag{6}$$

Considering that

$$G_n(\gamma^{n-1}\alpha) = G_{n+1}(\gamma^{n-1}\alpha^2) \cup [f^{-n}(H) \cap G_n(\gamma^{n-1}\alpha)] \cup [\cup_{j=1}^{\infty} S_{n+j}(\gamma^{n-1}\alpha\beta^{j})] \cup [\cap_{j=1}^{\infty} G_{n+j}(\gamma^{n-1}\beta^{j})],$$

(where $\mathrm{Leb}(\cap_{j=1}^{\infty} G_{n+j}(\gamma^{n-1}\beta^{j}) = 0)$, to prove (5) we only have to show that

$$\mathrm{Leb}(f^{(-n)}(H) \cap G_n(\gamma^{n-1}\alpha)) \geqslant \epsilon \mathrm{Leb}(G_n(\gamma^{n-1}\alpha)) + \sum_{j=1}^{\infty} \frac{\mathrm{Leb}(S_{n+j}(\gamma^{n-1}\alpha\beta^{j}))}{(1-\epsilon)^{j+1}} - \sum_{j=1}^{\infty} \mathrm{Leb}(S_{n+j}(\gamma^{n-1}\alpha\beta^{j})).$$

Using Lemma (15), assumption (3) and results from Section (2.2) we have that

$$\begin{aligned}
\mathrm{Leb}(f^{(-n)}(H) \cap G_n(\gamma^{n-1}\alpha)) &\geqslant \frac{1}{C_2} \mathrm{Leb}(H) \mathrm{Leb}(G_n(\gamma^{n-1}\alpha)) \\
&\geqslant \frac{1}{C_2} \left[\epsilon C_2 + C_2^2 \sum_{j=1}^{\infty} S_j(\beta^{j}) \left(\frac{1}{(1-\epsilon)^{j}} - 1 \right) \right] \mathrm{Leb}(G_n(\gamma^{n-1}\alpha)) \\
&\geqslant \epsilon \mathrm{Leb}(G_n(\gamma^{n-1}\alpha)) + \sum_{j=1}^{\infty} \frac{\mathrm{Leb}(S_{n+j}(\gamma^{n-1}\alpha\beta^{j}))}{(1-\epsilon)^{j}} \\
&\quad - \sum_{j=1}^{\infty} \mathrm{Leb}(S_{n+j}(\gamma^{n-1}\alpha\beta^{j})).
\end{aligned}$$

and the proof of (5) is concluded. Now we prove (6).

$$\sum_{j=1}^{\infty} \frac{\text{Leb}(S_{n+j-1}(\gamma^{n-1}\beta^j))}{(1-\epsilon)^j}(1-\epsilon) = \text{Leb}(S_n(\gamma^{n-1}\beta)) + \sum_{j=1}^{\infty} \frac{\text{Leb}(S_{n+j}(\gamma^{n-1}\beta^{j+1}))}{(1-\epsilon)^j}. \quad (7)$$

Since

$$G_{n+1}(\gamma^{n-1}\beta\alpha) \subset S_n(\gamma^{n-1}\beta)$$

the proof is complete. ∎

Proof of Lemma 13. There exists an ϵ such that $L_{n+1} \leqslant L_n(1-\epsilon)$; then the sequence L_n decays exponentially. From Eq. (2) it is clear that $\text{Leb}(\Lambda_n) \leqslant L_n$. Then $\text{Leb}(\Lambda_n) \leqslant \eta^n$ for some $\eta \in (0, 1)$ and all natural n. ∎

4. CONTINUITY OF THE HAUSDORFF DIMENSION: THE VOLUME COMPARISON METHOD

This section is dedicated to proving Theorem 3. Later, in Section 6, we will show how to extend this statement to the more general case of non-uniformly expanding maps with hole.

4.1. Ideas and Motivations on the Volume Comparison Method

While for fixing an upper estimate to limit capacity and Hausdorff dimension we just have to find a sequence of "efficient" coverings of the set, to establish a lower estimate it is necessary to show that there is no such sequence. More precisely, to prove that $HD(X) \geqslant \alpha$ we should find a constant κ so that any sequence $\mathcal{U}_n$ of coverings to X such that

$$\sup_{U_j \in \mathcal{U}_n} \text{diam} U_j \to 0 \text{ when } n \to \infty$$

satisfies

$$\sum_{U_j \in \mathcal{U}_n} (\text{diam} U_j)^\alpha > \kappa \text{ for all } n \text{ big enough.} \quad (8)$$

We name the method developed in this part of this article *volume comparison method*. The first step in our proof consists of showing that, instead of considering all the sequences of coverings, we can just look at a smaller class of coverings which we call square coverings, a sort of discretization of the set of coverings. Although this simplification allows us to consider only countable coverings, we still need to verify that each one of them satisfies property (8).

Next step in our method is to build a fractal set Φ whose Hausdorff dimension we know. We call this set Φ a regular fractal set. Then we show that given any square covering of our repeller, we can find a covering of this fractal set just changing the position of the elements of that covering. This implies that our repeller has Hausdorff dimension greater than the Hausdorff dimension of the regular fractal set.

4.2. Discretization of the Set of Coverings

We will use here definition 1 of square partitions.

Definition 3. If S is a set contained in a cube we call k-square covering of S any finite covering contained in the union of all k-square partitions of any order. That is each element of the covering is an n-element of k-square partition for some n.

We observe that two elements of a k-square covering either have disjoint interiors or one is contained in the other. In our arguments k will always be a fixed constant, so sometimes we will refer to square coverings omitting the k.

Lemma 18. In the definition of Hausdorff dimension for a compact set contained in a cube it is enough to consider only k-square coverings (for any k fixed).

Proof. Consider a compact set Λ with Hausdorff dimension h. Given $\alpha < h$ it is obvious that for any sequence $\mathcal{V}_n$ of k-square coverings to Λ whose diameters converge to zero when n tends to infinity we have that

$$\sum_{V_i \in \mathcal{V}_n} \operatorname{diam}(V_i)^{\alpha} \tag{9}$$

goes to infinity with n. We claim that if $\beta > h$ then there exists a sequence $\mathcal{V}_n$ of k-square coverings to Λ whose diameters converge to zero as n tends to infinity and such that $\sum_{V_i \in \mathcal{V}_n} \operatorname{diam}(V_i)^{\beta}$ goes to zero. Let us construct such a covering. Consider $\mathcal{U}_n$ a sequence of coverings to Λ such that

$$\sum_{U_i \in \mathcal{U}_n} \text{diam}(U_i)^\beta \to 0$$

as n tends to infinity (such a sequence exists by the definition of Hausdorff dimension). By compacity of Λ we can assume that $\mathcal{U}_n$ is finite for each n. Notice that if

$$1/k^{(j+1)} < \text{diam}(U) < 1/k^j,$$

U can be covered by 2^d elements of the k-square partition of order j. Then we replace U by those 2^d cubes. The result is a new sequence of partitions $\mathcal{V}_n$ such that

$$\sum_{V_i \in \mathcal{V}_n} \text{diam}(V_i)^\beta \leqslant 2^d k^\beta \sum_{U_i \in \mathcal{U}_n} \text{diam}(U_i)^\beta$$

The sequence $\mathcal{V}_n$ satisfies our claim. ∎

4.3. Regular Fractal Sets

Next step is to define the k-regular fractal set Φ, the intersection of sets Φ_n that we construct now. Again we consider k fixed and we use the term n-element meaning an element of the k-square partition of order n (recall Definition 1).

Definition 4. A k-regular fractal set is a set Φ that can be obtained as intersections of sets Φ_n that are, by their turn, built according to the following inductive procedure:

- Remove an 1-element from the unitary d-dimensional cube. The remaining set is Φ_1;
- For each Φ_l built, define a block of Φ_l as an intersection of Φ_l and an l-element;
- To obtain Φ_{l+1} remove from each block of Φ_l one $(l+1)$-element.

This procedure results in a sequence of nested sets Φ_n. Each Φ_n is the union of $(k^d - 1)^n$ blocks. A k-regular fractal set Φ is a set obtained as limit of such a sequence Φ_n. Figure 1 illustrate a set Φ_3 when $d=2$ and $k=3$. The set Φ obtained in this case, if we always remove the middle n-element from blocks of Φ_{n-1}, is known as Sierpinski carpet.

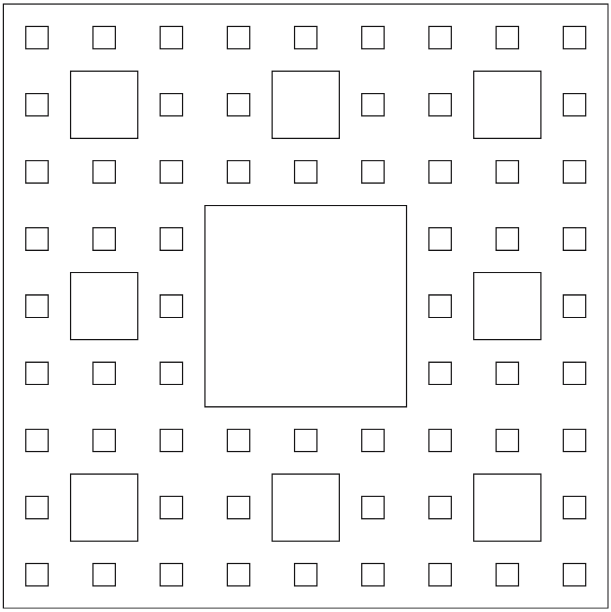

Fig. 1. Constructing Sierpinski carpet.

Proposition 19. The Hausdorff dimension of a k-regular fractal set Φ is $\log(k^d - 1)/\log k$.

Proof. Consider the function ρ assigning to each square covering $\mathcal{V} = \{V_1, \dots, V_n\}$

$$\rho(\mathcal{V}) = \sum_{j=1}^{n} \operatorname{diam}(V_j)^{\log_k(k^d - 1)}.$$

If we consider $\mathcal{V}_n$ the square covering of Φ whose elements are the blocks of Φ_n it is easy to verify that $\rho(\mathcal{V}_n) = 1$ for all n. This implies that the Hausdorff dimension of Φ is at most $\log(k^d - 1)/\log k$. On the other hand, we claim that any square covering $\mathcal{U}$ satisfies $\rho(\mathcal{U}) \geqslant 1$. By Lemma 18 this claim implies that the Hausdorff dimension of Φ is at least $\log(k^d - 1)/\log k$, so the lemma is proved except by the claim. So let us prove it.

To prove the claim we show that any square covering can be obtained by successive refinements of a square covering composed only by 1-elements, and that during the process of refinement ρ does not decrease.

Given a square covering $\mathcal{U} = \{U_1, \dots, U_l\}$ we don't lose generality assuming that all its elements intersect Φ. We claim that this assumption implies $\rho(\mathcal{U})=1$. Consider the covering $\mathcal{U}_1$ composed by the 1-elements that contain some $U_i \in \mathcal{U}$. Starting from $\mathcal{U}_1$ we construct by induction a sequence $\mathcal{U}_n$ of square coverings according to the following rule: $\mathcal{U}_n$ is the square covering composed by all the elements of $\mathcal{U}_{n-1}$ that coincides with some $U_i \in \mathcal{U}$ and all the n-elements that contain some $U_i \in \mathcal{U}$. There exists N such that $\mathcal{U}_n = \mathcal{U}$ for all $n \geqslant N$. Next two assertions are direct consequences of the definition of $\mathcal{U}_n$:

- Each $\mathcal{U}_n$ cover all the blocks of Φ_n;
- Each n-element of $\mathcal{U}_n$ is a block of Φ_n.

They clearly imply that $\rho(\mathcal{U}_1)=1$, and we use them to show that $\rho(\mathcal{U}_n)=\rho(\mathcal{U}_{n+1})$. Notice that if U_i is an n-element of $\mathcal{U}_n$ than either it is an element of $\mathcal{U}_{n+1}$ or it will be replaced by the (k^d-1) blocks of Φ_{n+1} contained in U_i. Since

$$\begin{aligned}(k^d-1)\left(\frac{1}{k^{n+1}}\right)^{\log_k(k^d-1)} &= (k^d-1)(k^d-1)^{-(n+1)}=(k^d-1)^{-n}\\ &= \left(\frac{1}{k^n}\right)^{\log_k(k^d-1)}\end{aligned}$$

it follows that in any case $\rho(\mathcal{U}_{n+1})=\rho(\mathcal{U}_n)$, what proves our claim. ∎

4.4. Comparing the Repeller with Regular Fractals

Now that we already know the Hausdorff dimension of k-regular fractals, we shall use this information to estimate by comparison the Hausdorff dimension of our repeller. Recall that we want to show that given $\alpha<d$, the Hausdorff dimension of Λ will be greater than α provided that the Lebesgue measure of the hole is small enough. So let us consider an $\alpha<d$ fixed. First step is to chose the k-regular set adequate to the comparison, that is, the correct k, depending on α. So now we fix k such that

$$\alpha<\frac{\log(k^d-1)}{\log k}. \tag{10}$$

Moreover let us fix an expansion constant σ satisfying

$$e^{-\sigma}<1/dk^{2d+1}. \tag{11}$$

Once fixed σ we choose N to assure that the F^N is σ-expanding (it is enough to consider N such that $e^{-cN} < e^{-\sigma}$). From now on we consider all these constants fixed and when we refer to a cylinder or the repeller it is always with respect to F^N. We also use H to mean H_{F^N} and Λ instead of Λ_{F^N}. Notice that the image domains W_i remain the same. We shall consider without loss of generality that ρ, the maximum inner diameter of W_i, is equal to 1 and we recall that $D = \inf\{\text{Leb}(W_i) : i = 1, \ldots, m\}$. Now we have a simple technical lemma useful while dealing with square coverings to Λ.

Lemma 20. Consider a k-square covering $\mathcal{P} = \{P_1^{\ell_1}, \ldots, P_m^{\ell_m}\}$ to Λ where P^{ℓ_i} is an element on level ℓ_i. Suppose $\ell_i \leqslant L$ for all i. So $\mathcal{P}$ cover Λ_l for all $l \geqslant L$.

Proof. The diameters of the cylinders in Λ_L are at most $e^{-\sigma l}$ (F^N is σ-expanding and $\rho = 1$), and the elements not covered by $\mathcal{P}$ are cubes whose sides are at minimum $1/k^L$. So considering that $1/k \geqslant e^{-\sigma}$ it follows that if there is a point of Λ_L not covered it must exist a point in the boundary of L-cylinder not covered, so $\mathcal{P}$ is not a covering of Λ (due to the Markovian property the boundary of Λ_L is contained in Λ). The result follows if we notice that $\Lambda_l \subset \Lambda_L$ for all $l \geqslant L$. ∎

Proposition 21. Consider the constants fixed above. If

$$\frac{\text{Leb}(H)}{D} < \frac{1}{C_2}\left(1 - \frac{k^d - 1}{k^d}\right), \tag{12}$$

where C_2 is the distortion constant to F^N(according to Section 2.2), then the Hausdorff dimension of Λ is bigger or equal than $\frac{\log(k^d-1)}{\log k}$

Proof. We show that given any k-square covering of Λ such that the levels of its elements are big enough we can use such covering to cover Φ, a k-regular fractal set, rearranging the elements of the covering. Let $\mathcal{U}$ be a square covering of Λ, and consider that its elements $U_1, U_2, \ldots, U_n$ have levels $\ell_1 \leqslant \ell_2 \leqslant \cdots \leqslant \ell_n$. We use X_j to represent $M \backslash (\cup_{i \leqslant j} U_i)$. We will construct a covering $\mathcal{U}' = \{U_1', \ldots, U_n'\}$ to Φ where the level of U_i' is ℓ_i for all $i \leqslant n$ (U_i and U_i' have the same level). We use X_j' to represent $\Phi \backslash (\cup_{i \leqslant j} U_i')$, analogously to X_j. The strategy is the following: notice that the volume of Λ_{ℓ_1} is bigger than the volume of Φ_{ℓ_1}, use U_1' to cover some part of Φ_{ℓ_1} and then observe again that if it remains something to be covered in Φ, the volume not covered in Φ_{ℓ_2} is smaller than $\text{Leb}(R_1 \cap \Lambda_{\ell_2})$, the volume not covered by U_1 in Λ_{ℓ_2}. We repeat this procedure until Φ be completely covered.

The volume of Φ_{ℓ_1} is $((k^d-1)/k^d)^{\ell_1}$ while the volume of Λ_{ℓ_1} is at least $\text{Leb}(\Lambda_1)(1-C_2\text{Leb}(H)/D)^{\ell_1}$ (by Corollary 9), so, if ℓ_1 is large enough, (12) implies that

$$\text{Leb}(\Phi_{\ell_1}) < \text{Leb}(\Lambda_{\ell_1}).$$

Take U_1', an ℓ_1-element, to cover any block of Φ_{ℓ_1}. Since

$$\text{Leb}(U_1') = \text{Leb}(\Phi_{\ell_1} \cap U_1') \geqslant \text{Leb}(\Lambda_{\ell_1} \cap U_1)$$

it follows that

$$\text{Leb}(\Phi_{\ell_1} \cap R_1') \leqslant \text{Leb}(\Lambda_{\ell_1} \cap R_1).$$

Suppose, as induction hypothesis, that for $1 \leqslant j \leqslant n$ we have chosen sets $U_1', \ldots, U_j'$ to cover Φ such that each U_i' is an ℓ_i-element and

$$\text{Leb}(\Phi_{\ell_j} \cap X_j') \leqslant \text{Leb}(\Lambda_{\ell_j} \cap X_j). \tag{13}$$

If $j=n$ the right hand side of last inequality is zero, and so the left too. In this case the proof is complete. Also if there are no more blocks to be covered in Φ_{ℓ_j}, we have already covered Φ. Let us consider that $j \neq n$ and we still have some uncovered blocks in Φ_{ℓ_j} and show that in this case we also have (13) with j replaced by $j+1$. We claim that:

$$\text{Leb}(\Phi_{\ell_j} \cap X_j') \leqslant \text{Leb}(\Lambda_{\ell_j} \cap X_j).$$

implies

$$\text{Leb}(\Phi_{\ell_{j+1}} \cap X_j') \leqslant \text{Leb}(\Lambda_{\ell_{j+1}} \cap X_j). \tag{14}$$

(That is, (13) $\Rightarrow$ (14).)

Let us finish the proof of the proposition assuming the claim. Chose a non-covered ℓ_{j+1} block in $\Phi_{\ell_{j+1}}$ and consider $U_{\ell_{j+1}}'$ as the respective element in the square partition of order ℓ_{j+1} . Once again we have

$$\text{Leb}(U_{j+1}') = \text{Leb}(\Phi_{\ell_{j+1}} \cap U_{j+1}') \geqslant \text{Leb}(\Lambda_{\ell_{j+1}} \cap U_{j+1})$$

what, since we assumed (14), implies

$$\text{Leb}(\Phi_{\ell_{j+1}} \cap X_{j+1}') \leqslant \text{Leb}(\Lambda_{\ell_{j+1}} \cap X_{j+1}) \tag{15}$$

completing the proof (since we have shown that the induction step can be performed), except by the claim.

To show the claim we first notice that if $\ell_j = \ell_{j+1}$, (14) is automatic, so we only have to consider the case $\ell_{j+1} > \ell_j$. If Φ_{ℓ_j} is not completely covered, it remains uncovered at least one element of the k-square partition of order ℓ_j, so

$$\mathrm{Leb}(X_j \cap \Lambda_{\ell_j}) \geqslant \mathrm{Leb}(X'_j \cap \Phi_{\ell_j}) \geqslant 1/k^{d\ell_j}.$$

We have to remove some pre-images of H from Λ_{ℓ_j} to find $\Lambda_{\ell_{j+1}}$, but we want to do it in such a way that we keep the control over $\mathrm{Leb}(X_j \cap \Lambda_{\ell_{j+1}})$. We can use bounded distortion arguments only to those cylinders that are contained in X_j. We use the condition on exponential decays of diameters of cylinders to ensure that the ℓ_j-cylinders intersecting the boundary of elements on the k-square partition of level ℓ_j are not representative in this context. The volume of ℓ_j-cylinders that intersect the boundary of the square partition of order ℓ_j is at most $d(ke^{-\sigma})^{\ell_j}$. Considering the assumption

$$e^{-\sigma} < \frac{1}{dk^{2d+1}}$$

we have that

$$d(ke^{-\sigma})^{\ell_j} \leqslant \frac{1}{d^{\ell_j} k^{2d\ell_j}} \leqslant \frac{1}{d^{\ell_j} k^{d\ell_j}} \mathrm{Leb}(X_j \cap \Lambda_{\ell_j})$$

It follows that the volume of the cylinders in Λ_{ℓ_j} that are not contained in X_j is bounded by a small fraction of $\Lambda_{\ell_j} \cap X_j$ and the remaining part, strictly contained in X_j, has volume bigger or equal than

$$\mathrm{Leb}(X_j \cap \Lambda_{\ell_j}) - d(ke^{-\sigma})^{\ell_j} \geqslant \frac{d^{\ell_j - 1} k^{d\ell_j} - 1}{d^{\ell_j - 1} k^{d\ell_j}} \mathrm{Leb}(X_j \cap \Lambda_{\ell_j})$$

Now we have that

$$\mathrm{Leb}(X_j \cap \Lambda_{\ell_{j+1}}) \geqslant \frac{d^{\ell_j - 1} k^{d\ell_j} - 1}{d^{\ell_j - 1} k^{d\ell_j}} \mathrm{Leb}(X_j \cap \Lambda_{\ell_j}) \left(1 - C_2 \frac{\mathrm{Leb}(H)}{D}\right)$$

while

$$\mathrm{Leb}(X'_j \cap \Phi_{\ell_{j+1}}) = \mathrm{Leb}(X'_j \cap \Phi_{\ell_j}) \frac{k^d - 1}{k^d}$$

If ℓ_j is big enough we have (by hypothesis)

$$\frac{d^{\ell_j-1}k^{d\ell_j}-1}{k^{d\ell_j}}\left(1-C_2\frac{\mathrm{Leb}(H)}{D}\right)\geqslant\frac{d^{\ell_j-1}k^d-1}{k^d}$$

what implies the inequality (14):

$$\mathrm{Leb}(X_j\cap\Lambda_{\ell_{j+1}})\geqslant\mathrm{Leb}(X'_j\cap\Phi_{\ell_{j+1}}).\quad\blacksquare$$

Now we prove the Theorem 3.

Proof of Theorem 3. Notice that the repeller Λ_F is the same repeller Λ we had for F^N. Then $HD(\Lambda_F)=HD(\Lambda)$. We recall that N may be taken as the smaller natural number such that $e^{-cN}>e^{-\sigma}$. It is important to notice that N is a number that depends only on k, d and c.

We have to be careful because H_F is not the same H from last proposition. However, bounded distortion (more precisely, Corollary 9) ensures that

$$1-\mathrm{Leb}(H)\geqslant\left(1-C_2\frac{\mathrm{Leb}(H_F)}{D}\right)^N,$$

that is,

$$\mathrm{Leb}(H)\leqslant 1-\left(1-C_2\frac{\mathrm{Leb}(H_F)}{D}\right)^N.$$

By Proposition 21, to ensure that $HD(\Lambda)<\frac{\log(k^d-1)}{k}$ it is enough to have

$$1-\left(1-C_2\frac{\mathrm{Leb}(H_F)}{D}\right)^N<\frac{D}{C_2}\left(1-\frac{k^d-1}{k^d}\right).$$

Define the function $\kappa(x)$ as the infimum over all k that realize the inequality

$$1-\left(1-C_2\frac{x}{D}\right)^N<\frac{D}{C_2}\left(1-\frac{k^d-1}{k^d}\right).$$

or $\kappa(x)=1$ if there is not such k. It is clear that $\kappa(x)$ goes to infinity when x goes to zero. Now we just have to define

$$\psi_1(x)=\frac{\log(\kappa(x)^d-1)}{\log\kappa(x)}$$

($\psi_1(x)=0$ if $\kappa(x)=1$), and the proof is finished. $\blacksquare$

5. CONTINUITY OF THE LIMIT CAPACITY

In this section we prove Theorem 2. Later, in Section 6, we will show how to extend this statement to the more general case of non-uniformly expanding maps with hole.

By the well-known inequality $HD(X) \leqslant Cap(X)$ this theorem is a corollary from Theorem 3. On the other hand, as we shall see, the limit capacity is not so dependent on volume distribution as Hausdorff dimension. Some of the efforts involved in bypassing non-conformality when dealing with Hausdorff dimension, turn out not to be necessary for limit capacity. This is substantiated by the the fact that in the later context the proof is much shorter (about one page). The development of this shorter proof is based on volume control through the steps of construction of the repeller.

Proof of Theorem 2. It follows from the c-expanding assumption that the volume of cylinders have exponential decay:

$$\mathrm{Leb}(C_n) \leqslant e^{-nc}$$

for all n-cylinder C_n.

On the other hand, considering the distortion constant C_2 obtained in Section 2.2, we have the following recurrence relation

$$\mathrm{Leb}(\Lambda_n) \geqslant \mathrm{Leb}(\Lambda_{n-1}) \left(1 - C_2 \frac{\mathrm{Leb}(H_F)}{D}\right)$$

which implies that

$$\mathrm{Leb}(\Lambda_n) \geqslant \frac{(1 - \mathrm{Leb}(H_F))}{(1 - C_2 \frac{\mathrm{Leb}(H_F)}{D})} \left(1 - C_2 \frac{\mathrm{Leb}(H_F)}{D}\right)^n.$$

For k large enough we can fix a natural number q such that

$$k^{d+1} > e^{cq} > k^d. \tag{16}$$

Given a cylinder C_{nq} that intersects an element of the k-square partition of order n, we claim that there is a point of the boundary of C_{nq} inside the n-element. Indeed the volume of the n-element is bigger than the volume of any cylinder C_{nq}, what proves our claim. If N_n is the number of elements of the k-square partition of order n intersecting Λ_{nq} we have that

$$N_n \geqslant \frac{(1-\mathrm{Leb}(H_F))}{(1-C_2\frac{\mathrm{Leb}(H_F)}{D})}\left(1-C_2\frac{\mathrm{Leb}(H_F)}{D}\right)^{nq} k^{nd}$$
$$=\gamma\left[\left(1-C_2\frac{\mathrm{Leb}(H_F)}{D}\right)^{q} k^{d}\right]^{n}, \tag{17}$$

where γ does not depend on n. As we saw, in each of these N_n elements, there must be a point of the boundary of Λ_{nq} (thus a point of Λ_F). It is easy to see that among these N_n points of Λ_F we can chose $\frac{N_n}{3^d}$ points such that the distance between any two of them is bigger or equal to k^{-n}. Consequently, any covering of Λ_F by k^{-n}-balls has at least $\frac{N_n}{3^d}$ elements. It follows that

$$\mathrm{Cap}(\Lambda_F) \geqslant -\limsup_{n\to\infty} \frac{\log(N_n/3^d)}{\log k^{-n}}.$$

Considering (17),

$$\mathrm{Cap}(\Lambda_F) \geqslant \frac{\log[(1-C_2\frac{\mathrm{Leb}(H_F)}{D})^q k^d]}{\log k} = d + q\frac{\log(1-C_2\frac{\mathrm{Leb}(H_F)}{D})}{\log k}.$$

Now it is clear that if $\mathrm{Leb}(H_F)$ tends to zero, $\mathrm{Cap}(\Lambda_F)$ tends to d. Moreover, considering (16), the function

$$\psi_0(x) = \max\left\{d + \frac{d+1}{c}\log\left(1-C_2\frac{x}{D}\right), 0\right\}$$

satisfies the statement of the theorem. ∎

6. FROM NON-UNIFORM TO UNIFORM EXPANSION

This section aims to provide results that allow us to extend the theorems from last sections to the case of non-uniformly expanding maps. This is done through Theorem 4. The goal is to use a map f satisfying (NU_1) such that $\log|\det(Df^{(-1)})|$ is (C_0, ϵ)-Hölder for any inverse branch $f^{(-1)}$ of f to construct an induced map F c-expanding such that $\mathrm{Leb}(H_F)$ is so small as we wish, provided that so does $\mathrm{Leb}(H_f)$. Furthermore F shall be such that $\log|\det(DF^{(-1)})|$ is (C_0', ϵ)-Hölder for any inverse branch $F^{(-1)}$ of F and the constants C_0' shall depend only on C_0, ϵ and c.

Proof of Theorem 4. Let us consider the map F_n such that for all $j \leqslant n$, $F_n(x) = f^j(x)$ if $x \in \mathcal{B}_{j-1} \backslash \mathcal{B}_j$. In the set $\mathcal{B}_n$ we do not define F_n, we consider this set as part of H_{F_n}. Therefore we have

$$H_{F_n} = \mathcal{B}_n \cup (f^{-1}(H_f) \cap \mathcal{B}_1) \cup \cdots \cup (f^{-(n-1)}(H_f) \cap \mathcal{B}_{n-1}).$$

Defining $T = \max\{1, S^d\}$, , we claim that

$$\text{Leb}(H_{F_n}) \leqslant \delta_n + \text{Leb}(H_f) \sum_{j=0}^{n} m^j T^j$$

Indeed $\text{Leb}(\mathcal{B}_n) \leqslant \delta_n$ and $\text{Leb}(f^{-j}(H_f) \cap \mathcal{B}_j)$ is bounded by the amount of cylinders in $\mathcal{B}_j$ times the maximum Lebesgue measure of $f^{-j}(H_f)$ for each inverse branch. The amount of cylinders is bounded by m^j and $\text{Leb}(f^{-j}(H_f)) \leqslant T^j \text{Leb}(H_f)$.

Consider the function $\psi : [0, 1] \to R$ defined by

$$\psi(x) = \inf_{n \in \mathbb{N}} \left(\delta_n + x \sum_{j=1}^{n} m^j T^j \right)$$

We noticed that the infimum is attained for some n (the expression is increasing after some n). Given f we will consider as F the map F_n such that n is the natural that realizes the infimum $\psi(\text{Leb}(H_f))$. We define $\psi_2(x) = min\{1, \psi(x)\}$. It is clear that the map F and the function ψ_2 so defined satisfy the assertion of the theorem. ▮

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